TA Dispersion in AF4

$$\frac{\partial c}{\partial t} - u_p \frac{\partial c}{\partial y} + \dot{\gamma} y \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2}$$

$$y^* = y/y_c \qquad x^* = \frac{x}{x_c} \qquad t^* = t/t_c \qquad c^* = \frac{c}{c_c}$$

$$-u_p c\Big|_{y=0} - D\frac{\partial c}{\partial y}\Big|_{y=0} = 0$$

$$\therefore \quad y_c = \frac{D}{u_p}$$

$$\int_{-\infty}^{\infty}\int_0^{\infty} c \, dy \, dx = I \qquad (\text{conservation})$$

$$\left[\frac{y_c^2}{D t_c}\right]\frac{\partial c^*}{\partial t^*} - \frac{\partial c^*}{\partial y^*} + \left[\frac{\dot{\gamma} y_c^3}{D x_c}\right] y^* \frac{\partial c^*}{\partial x^*} = \frac{\partial^2 c^*}{\partial y^{*2}}$$

$$t_c = \frac{y_c^2}{D} = \frac{D}{u_p^2} \qquad x_c = \frac{\dot{\gamma} y_c^3}{D} = \frac{\dot{\gamma} D^2}{u_p^3}$$

$$c_c = \frac{I}{x_c y_c}$$

So: $$\frac{\partial c^*}{\partial t^*} - \frac{\partial c^*}{\partial y^*} + y^* \frac{\partial c^*}{\partial x^*} = \frac{\partial^2 c^*}{\partial y^{*2}}$$

$$c^*\Big|_{y^* \to \infty} = 0 \qquad \frac{\partial c^*}{\partial y^*}\Big|_{y^*=0} + c^*\Big|_{y^*=0} = 0$$

We define:

$$c_p^* = \int_{-\infty}^{\infty} x^{*p} c^* \, dx^*$$

$$m_p^* = \int_{0}^{\infty} c_p^* \, dy^*$$

So:

$$\frac{\partial c_p^*}{\partial t^*} - \frac{\partial c_p^*}{\partial y^*} + \int_{-\infty}^{\infty} y^* \frac{\partial c^*}{\partial x^*} x^{*p} \, dx^* = \frac{\partial^2 c_p^*}{\partial y^{*2}}$$

Now $\displaystyle\int_{-\infty}^{\infty} y^* \frac{\partial c^*}{\partial x^*} x^{*p} \, dx^* = -p \, y^* c_{p-1}^*$

So:

$$\frac{\partial c_p^*}{\partial t^*} = p \, y^* c_{p-1}^* + \frac{\partial c_p^*}{\partial y^*} + \frac{\partial^2 c_p^*}{\partial y^{*2}}$$

and

$$\frac{d m_p^*}{d t^*} = p \int_{0}^{\infty} y^* c_{p-1} \, dy^* + \underbrace{\left[c_p^* + \frac{\partial c_p^*}{\partial y^*} \right]_0^1}_{=0}$$

so $\displaystyle\frac{d m_p^*}{d t^*} = p \int_0^{\infty} y^* c_{p-1} \, dy^*$

We have the sequence:

$p=0$ $\quad \dfrac{\partial m_0^*}{\partial t^*} = 0 \quad \therefore \quad m_0^* = 1$

$$\frac{\partial c_0^*}{\partial t^*} = \frac{\partial c_0^*}{\partial y^*} + \frac{\partial^2 c_0^*}{\partial y^{*2}} \qquad \int_0^\infty c_0^* \, dy^* = m_0^* = 1$$

The BC is $c_0\Big|_{y^* \to \infty} = 0, \quad c_0\Big|_{y^*=0} + \dfrac{\partial c_0^*}{\partial y^*}\Big|_{y^*=0} = 0$

$$\therefore \quad \underline{c_0^* = e^{-y^*}}$$

$p=1$ $\quad \dfrac{\partial m_1^*}{\partial t^*} = \displaystyle\int_0^\infty y^* c_0^* \, dy^* = \int_0^\infty y^* e^{-y^*} \, dy^*$

$$= \underbrace{-y^* e^{-y^*}\Big|_0^\infty}_{0} + \int_0^\infty e^{-y^*} \, dy^* = 1$$

so $\underline{m_1^* = t^*}$

$$\frac{\partial c_1^*}{\partial t^*} = y^* c_0^* + \frac{\partial c_1^*}{\partial y^*} + \frac{\partial^2 c_1^*}{\partial y^{*2}}$$

Let $C_1^* = t^* C_0^* + f(y^*)$

$$= t^* e^{-y^*} + f(y^*)$$

$$\therefore\; e^{-y^*} = y^* e^{-y^*} + t^* \underbrace{\left(-e^{-y^*} + e^{-y^*}\right)}_{0} + f' + f''$$

So $f'' + f' = (1 - y^*) e^{-y^*}$

We can integrate this:

$$f' + f = e^{-y}\left(y^* - \tfrac{1}{2} y^{*2} - 1\right)$$

where the constant is zero from the BC.

So:

$$f = e^{-y^*}\left[\int \cancel{e^{-y^*}}\left(y^* - \tfrac{1}{2} y^{*2} - 1\right)\cancel{e^{y^*}}\, dy^* + cst\right]$$

$$= e^{-y^*}\left(\tfrac{1}{2} y^{*2} - \tfrac{1}{6} y^{*3} - y^* + cst\right)$$

Now $\int_0^\infty f\, dy^* = 0$

$$\therefore\; cst = -\int_0^\infty e^{-y^*}\left(\tfrac{1}{2} y^{*2} - \tfrac{1}{6} y^{*3} - y^*\right) dy^*$$

$= 1\,!$

So $f = e^{-y^*}\left(1 - y^* + \frac{1}{2}y^{*2} - \frac{1}{6}y^{*3}\right)$

and $c_1^* = t^* e^{-y^*} + f(y^*)$

Finally, $\frac{dm_2^*}{dt^*} = 2\int_0^\infty y^* c_1\, dy^*$

so $\frac{\kappa}{D} = \left[\frac{x_c^2}{D t_c}\right] \int_0^\infty y^* f(y^*) dy^*$

$= \frac{x_c^2}{D t_c} \int_0^\infty e^{-y^*}\left(y^* - y^{*2} + \frac{1}{2}y^{*3} - \frac{1}{6}y^{*4}\right) dy^*$

$\underline{= 2}$

$t_c = \frac{y_c^2}{D}$

$\frac{x_c^2}{D t_c} = \frac{(\dot{\gamma} y_c t_c)^2}{D t_c} = \frac{\dot{\gamma}^2 y_c^2 t_c}{D} = \frac{\dot{\gamma}^2 y_c^4}{D^2}$

$y_c = \frac{D}{u_0} \quad \therefore \quad \frac{x_c^2}{D t_c} = \frac{\dot{\gamma}^2 D^2}{u_0^4}$

so $\frac{\kappa}{D} = 2 \frac{\dot{\gamma}^2 D^2}{u_0^4}$